Strict Topologies and Vector-Measures on non-Archimedean Spaces

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Introduction

Let $C_b(X,E)$ be the space of all bounded continuous functions from a zero-dimensional Hausdorff topological space X to a non-Archimedean Hausdorff locally convex space E. In section 2 of this paper, we look at some of the properties of the locally convex topologies β, β', β_1 and β'_1 on $C_b(X,E)$, introduced by the author in [8], and we show that the corresponding dual spaces are certain subspaces of a space $M(X,E')$ of finitely-additive E'-valued measures on the algebra of all clopen subsets of X introduced in [6]. In case E is a polar space, it is proved that the strict topology β_o, which was defined by the author in [7], coincides with the polar topology associated with β'. In section 3 we look at the supports of members of $M(X,E')$ and in section 4 we introduce the topologies β_e and β'_e. In case E is metrizable, it is shown that β_e is coarser than β_1 and coincides with the topology of simple convergence on uniformly bounded equicontinuous subsets of $C_b(X,E)$. In section 5 we look at the dual spaces of $C_b(X,E)$ under the topologies β_u and β'_u, which were defined in [1] and [3], respectively, while in section 6 we investigate the dual spaces for the topologies β_e and β'_e. When E is metrizable, it is proved that β'_e yields as dual space the space of the so called separable members of $M(X,E')$ and that the same does β'_u. Moreover the two topologies have the same equicontinuous sets in their common dual space.

1 Preliminaries

Throughout this paper, $\mathbb{K}$ stands for a complete non-Archimedean valued field whose valuation is non-trivial. By a seminorm, on a vector space E over $\mathbb{K}$, we mean a non-Archimedean seminorm. Similarly, by a locally convex space we mean a non-Archimedean locally convex space over $\mathbb{K}$. For E a locally convex space, we denote by $cs(E)$ the collection of all continuous seminorms on E and by E' its dual space.

Let now X be a zero-dimensional Hausdorff topological space and E a Hausdorff locally convex space. We will denote by $\beta_o X$ the Banaschewski compactification of X (see [4]) and by $\upsilon_o X$ the $\mathbb{N}$-repletion of X ($\mathbb{N}$ is the set of natural numbers), i.e. the subspace of $\beta_o X$ consisting of all $x \in \beta_o X$ with the following property: For each sequence (V_n) of neighborhoods of x in $\beta_o X$ we have that $\bigcap V_n \cap X \neq \emptyset$. The space X is called $\mathbb{N}$-replete if $X = \upsilon_o X$. We will denote by $C_b(X, E)$ the space of all bounded continuous E-valued functions on X and by $C_{rc}(X, E)$ the space of all $f \in C_b(X, E)$ for which $f(X)$ is relatively compact in E. In case $E = \mathbb{K}$, we will simply write $C_b(X)$ and $C_{rc}(X)$ respectively. For $A \subset X$, we denote by χ_A the $\mathbb{K}$-valued characteristic function of A in X and by $\overline{A}^{\beta_o X}$ the closure of A in $\beta_o X$. Every $f \in C_{rc}(X, E)$ has a unique continuous extension f^{β_o} to all of $\beta_o X$. For f an E-valued function on X, p a seminorm on E and $A \subset X$, we define

$$\|f\|_p = \sup_{x \in X} p(f(x)), \quad \|f\|_{A,p} = \sup_{x \in A} p(f(x)).$$

The strict topology β_o on $C_b(X, E)$ (see [7]) is the locally convex topology generated by the seminorms $f \mapsto \|hf\|_p$, where $p \in cs(E)$ and h is in the space $B_o(X)$ of all bounded $\mathbb{K}$-valued functions on X which vanish at infinity, i.e. for each $\epsilon > 0$ there exists a compact subset Y of X such that $|h(x)| < \epsilon$ if x is not in Y. As it is shown in [7], β_o has the same bounded sets with the topology τ_u of uniform convergence, i.e. the topology generated by the seminorms $\|.\|_p, p \in cs(E)$. Also β_o coincides with the topology τ_k of compact convergence on τ_u-bounded subsets of $C_b(X, E)$.

Let now $K(X)$ be the algebra of all clopen, (i.e. closed and open) subsets of X. We denote by $M(X, E')$ (see [6]) the space of all finitely-additive E'-valued measures m on $K(X)$ for which $m(K(X))$ is an equicontinuous subset of E'. For each m in $M(X, E')$ there exists $p \in cs(E)$ with $m_p(X) < \infty$, where, for $A \in K(X)$,

$$m_p(A) = \sup\{|m(B)s|/p(s) : p(s) \neq 0, A \supset B \in K(X)\}.$$

The space of all $m \in M(X, E')$ with $m_p(X) < \infty$ is denoted by $M_p(X, E')$. Next, we recall the definition of the integral of an E-valued function f on X with respect to an $m \in M(X, E')$. For $A \in K(X), A \neq \emptyset$, let $\mathcal{D}_A$ denote the family of all $\alpha = \{A_1, \ldots, A_n : x_1, \ldots, x_n\}$, where $\{A_1, \ldots, A_n\}$ is a clopen partition of A and $x_i \in A_i$. We make $\mathcal{D}_A$ a directed set by defining $\alpha_1 \geq \alpha_2$ iff the partition of A in α_1 is a refinement of the one in α_2. For $f \in E^X, m \in M(X, E')$ and $\alpha = \{A_1, \ldots, A_n : x_1, \ldots, x_n\}$, we define $\omega_\alpha(f, m) = \sum_{i=1}^n m(A_i) f(x_i)$. If the $\lim_\alpha \omega_\alpha(f, m)$ exists in $\mathbb{K}$, we will say that f is m-integrable over A and denote this limit by $\int_A f dm$. We define the integral over the empty set to be 0. For $A = X$, we write simply $\int f dm$. It is easy to see that if f is m-integrable over X, then it is m-integrable over every $A \in K(X)$ and $\int_A f dm = \int \chi_A f dm$. Every $m \in M(X, E')$ defines a τ_u-continuous linear functional on $C_{rc}(X, E)$ by $f \mapsto \int f dm$ (see [6]). Also every $\phi \in (C_{rc}(X, E), \tau_u)'$ is given in this way by a unique m.

For $p \in cs(E)$, we denote by $M_{t,p}(X, E')$ the space of all $m \in M_p(X, E')$ for which m_p is tight, i.e. for every $\epsilon > 0$, there exists a compact subset Y of X such that $m_p(A) \leq \epsilon$ if A is disjoint from Y. We define

$$M_t(X, E') = \bigcup_{p \in cs(E)} M_{t,p}(X, E').$$

As it is shown in [7], every $m \in M_t(X, E')$ defines a β_o-continuous linear form on $C_b(X, E)$ by $u_m(f) = \int f dm$. Moreover the map $m \mapsto u_m$, from $M_t(X, E')$ to $(C_b(X, E), \beta_o)'$, is an algebraic isomorphism. Finally we recall that a locally convex space E has the countable neighborhood property if, for each sequence (p_n) of continuous seminorms on E, there exist a $p \in cs(E)$ and a sequence (α_n) of positive numbers such that $p \geq \alpha_n p_n$ for all n. For all unexplained terms on locally convex spaces, we refer to [15] and [16].

Throughout the paper, X is a zero-dimensional Hausdorff topological space and E a Hausdorff locally convex space.

2 On the Topologies $\beta, \beta', \beta_1, \beta_1'$

We recall the definitions of the locally convex topologies β, β', β_1 and β_1' on $C_b(X, E)$ introduced by the author in [8]. Let $\Omega = \Omega(X)$ be the family of all compact subsets of $\beta_o X$ which are disjoint from X. For $H \in \Omega$, let C_H be the space of all $h \in C_{rc}(X)$ whose the continuous extension h^{β_o} vanishes on H. For $p \in cs(E)$, let $\beta_{H,p}$ denote the locally convex topology on $C_b(X, E)$ generated by the seminorms $\|.\|_{h,p}$, where $h \in C_H$ and $\|f\|_{h,p} = \|hf\|_p$. The inductive limit of the topologies $\beta_{H,p}$, as H ranges over Ω, is denoted by β_p, while β' is the projective limit of the topologies β_p, $p \in cs(E)$. Also, for $H \in \Omega$, β_H is the locally convex topology generated by the seminorms $\|.\|_{h,p}, h \in C_H, p \in cs(E)$. The inductive limit of the topologies $\beta_H, H \in \Omega$, is denoted by β. Replacing Ω by the family Ω_1 of all $\mathbb{K}$-zero subsets of $\beta_o X$ which are disjoint from X, we get the topologies $\beta_{1,p}$, for $p \in cs(E), \beta_1$ and β_1'. Recall that a $\mathbb{K}$-zero subset of $\beta_o X$ is a set of the form $\{x \in \beta_o X : g(x) = 0\}$ for some $g \in C(\beta_o X)$. Analogous with topologies β and β' are the topologies β_u and β_u' which were defined in [3]. They are obtained by replacing Ω by the family Ω_u of all $Q \in \Omega$ with the following property: There exists a clopen partition $(A_i)_{i\in I}$ of X with $\overline{A_i}^{\beta_o X}$ disjoint from Q for all $i \in I$.

Theorem 2.1 [8]. *For $H \in \Omega$ and $p \in cs(E)$, $\beta_{H,p}$ has as a base at zero the sets of the form*

$$\bigcap_{n=1}^{\infty} \{f \in C_b(X, E) : \|f\|_{A_n,p} \leq \alpha_n\},$$

where (α_n) is an increasing sequence of positive numbers, tending to ∞, and (A_n) an increasing sequence of clopen subsets of X with $\overline{A_n}^{\beta_o X}$ disjoint from H for all n.

We only sketch the proof of the next Theorem since it is a modification of the proof of Theorem 4.1 in [8].

Theorem 2.2 *An absolutely convex subset V of $C_b(X, E)$ is a $\beta_{H,p}$-neighborhood of zero iff the following condition is satisfied: For each $r > 0$, there exists a clopen subset A of X, with $\overline{A}^{\beta_o X}$ disjoint from H, and $\epsilon > 0$ such that*

$$\{f \in C_b(X, E) : \|f\|_p \leq r, \|f\|_{A,p} \leq \epsilon\} \subset V.$$

Proof: The necessity follows using the preceding Theorem. Conversely, suppose that the condition is satisfied and let $\lambda \in \mathbb{K}, |\lambda| > 1$. Choose an increasing sequence (A_n) of clopen sets, with $\overline{A_n}^{\beta_o X}$ disjoint from H, and a decreasing sequence (ϵ_n) of positive numbers, $\epsilon_n \to 0$, such that $U_n \cap \lambda^n U \subset V$, where

$$U_n = \{f \in C_b(X, E) : \|f\|_{A_n,p} \leq \epsilon_n\}, \quad U = \{f \in C_b(X, E) : \|f\|_p \leq 1\}.$$

Let $V_1 = U_1 \bigcap [\bigcap_{n=1}^{\infty}(U_{n+1} + \lambda^n U)]$. Then $V_1 \subset V$. Choose $\lambda_1 \in \mathbb{K}$, with $0 < \lambda_1| < \min\{1, \epsilon_1\}$, and take $\lambda_n = \lambda^{n-1}$ for $n > 1$. Now

$$\bigcap_{n=1}^{\infty}\{f \in C_b(X,E) : \|f\|_{A_n,p} \leq |\lambda_n|\} \subset V_1,$$

and hence the result follows from the preceding Theorem.

Corollary 2.3 *If $\tau_{u,p}$ is the topology generated by the seminorm $\|.\|_p$, then β_p is the finest loally convex topology on $C_b(X,E)$ which coincides with β_p on $\tau_{u,p}$-bounded sets.*

We will show next that the dual space of $C_b(X,E)$, under the topology β, is a certain subspace of $M(X,E')$. Let $M_\tau(X,E')$ be the space of all $m \in M(X,E')$ with the following property: For each net (A_δ) of clopen subsets of X which decreases to the empty set, there exists $p \in cs(E)$, with $m_p(X) < \infty$, such that $m_p(A_\delta) \to 0$. Replacing decreasing nets by decreasing sequences, we get the space $M_\sigma(X,E')$

Theorem 2.4 *If $m \in M_\tau(X,E')$, then every member of $C_b(X,E)$ is m-integrable and the linear map $u_m : C_b(X,E) \to \mathbb{K}, u_m(f) = \int f dm$, is β-continuous.*

Proof: There exists $p \in cs(E)$ with $m_p(X) \leq 1$. Let $f \in C_b(X,E)$ and $\epsilon > 0$. We may assume that $\|f\|_p \leq 1$. Let $(A_i)_{i\in I}$ be the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $p(f(x) - f(y)) \leq \epsilon$. Choose $x_i \in A_i$. The function $f^* = \sum_i \chi_{A_i} f(x_i)$ is continuous. For each finite subset J of I, set $B_J = \bigcup_{i\notin J} A_i$. Then B_J is clopen and $B_J \downarrow \emptyset$. By our hypothesis, there exists $q \in cs(E), q \geq p$, such that $m_q(B_J) \to 0$. Choose a finite subset J of I such that $m_q(B_J) < \epsilon/\|f\|_q$. Let $g = \sum_{i\in J} \chi_{A_i} f(x_i)$ and $h = f^* - g$. For any clopen partitioin $\{D_1, \dots, D_n\}$ of X, which is a refinement of $\{A_i \in J\} \cup \{B_J\}$, and any $y_k \in D_k$, we have

$$\left|\sum_{k=1}^{n} m(D_k)h(y_k)\right| \leq \epsilon \text{ and } \sum_{k=1}^{n} m(D_k)g(y_k) = \sum_{i\in J} m(A_i)f(x_i).$$

Thus

$$\left|\sum_{k=1}^{n} m(D_k)f^*(y_k) - \sum_{i\in J} m(A_i)f(x_i)\right| \leq \epsilon \text{ and } \left|\sum_{k=1}^{n} m(D_k)[f(y_k) - f^*(y_k)]\right| \leq \epsilon,$$

and so $\left|\sum_{k=1}^{n} m(D_k)f(y_k) - \sum_{i\in J} m(A_i)f(x_i)\right| \leq \epsilon$. It follows that f is m-integrable. Finally, u_m is β-continuous. Indeed, let $H \in \Omega$. It suffices to show that u_m is $\beta_{H,p}$-continuous for some $p \in cs(E)$. To this end, we first observe that there exists a decreasing net (B_δ) of clopen subsets of X with $\bigcap \bar{B_\delta}^{\beta_o X} = H$. Since $m \in M_\tau(X,E')$, there exists $p \in cs(E)$ such that $m_p(X) \leq 1$ and $\lim m_p(B_\delta) = 0$. We will show that u_m is $\beta_{H,p}$-continuous. Let $W = \{f \in C_b(X,E) : |u_m(f)| \leq 1\}$ and $r > 0$. There exists δ with $m_p(B_\delta) < 1/r$. If $B = X \setminus B_\delta$, then $\bar{B}^{\beta_o X}$ is disjoint from H and

$$\{f \in C_b(X,E) : \|f\|_p \leq r, \|f\|_{B,p} \leq 1\} \subset W.$$

The result now follows from Theorem 2.2.

Theorem 2.5 *The map $u : M_\tau(X,E') \to (C_b(X,E),\beta)', m \mapsto u_m$, is an algebraic isomorphism.*

Proof: It remains only to show that u is onto. So, let ϕ a β-continuous linear functional on $C_b(X,E)$. Since β is coarser that τ_u, there exists $m \in M(X,E')$ such that $\phi(f) = \int f dm$ for all $f \in C_{rc}(X,E)$. We will show that $m \in M_\tau(X,E')$. In fact, let (A_δ) be a net of clopen sets, which decreases to the empty set, and let $H = \bigcap \bar{A_\delta}^{\beta_o X}$. Since ϕ is β_H-continuous, there exist $p \in cs(E)$ and $h \in C_H$ such that

$$W_1 = \{f \in C_b(X,E) : \|hf\|_p \le 1\} \subset \{f : |\phi(f)| \le 1\}.$$

We will show that $m_p(A_\delta) \to 0$. So, let μ be a non-zero element of $\mathbb{K}$. The set

$$G = \{x \in \beta_o X : |h^{\beta_o}(x)| \le |\mu|\}$$

is clopen and contains H. There exists δ with $\bar{A_\delta}^{\beta_o X} \subset G$. If now A is a clopen subset of A_δ and $s \in E$ with $p(s) \le 1$, then $\mu^{-1}\chi_A s \in W_1$ and so $|m(A)s| \le |\mu|$. If $\lambda \in \mathbb{K}$ with $|\lambda| \le 1$, then $m_p(A_\delta) \le |\lambda\mu|$, which clearly proves that $m \in M_\tau(X,E')$. Finally, $\phi = u_m$ since both ϕ and u_m are β-continuous and coincide on the β-dense subset $C_{rc}(X,E)$ of $C_b(X,E)$.

Using arguments analogous to the ones used in the proofs of Theorems 2.4 asnd 2.5, we get the following

Theorem 2.6 *A subset H, of the dual space $M_\tau(X,E')$ of $(C_b(X,E),\beta)$, is β-equicontinuous iff the following condition is satisfied: For each net (A_δ) of clopen subsets of X, which decreases to the empty set, there exists $p \in cs(E)$ such that $\sup_{p\in H} m_p(X) < \infty$ and $\sup_{m\in H} m_p(A_\delta) \to 0$.*

Next we will look at the dual space of $(C_b(X,E),\beta')$. For $p \in cs(E)$, let $\mathcal{M}_{\tau,p}(X,E')$ be the space of all $m \in M_p(X,E')$ such that $m_p(A_\delta) \to 0$ for each net (A_δ) of clopen subsets of X which decreases to the empty set. Let

$$\mathcal{M}_\tau(X,E') = \bigcup_{p\in cs(E} \mathcal{M}_{\tau,p}(X,E').$$

Replacing nets by decreasing sequences of clopens sets, we get the spaces $\mathcal{M}_{\sigma,p}(X,E')$ and $\mathcal{M}_\sigma(X,E')$.

As in the proofs of Theorems 2.4 and 2.5, we get that, for $m \in \mathcal{M}_{\tau,p}(X,E')$, u_m is β_p-continuous and every $\phi \in (C_b(X,E),\beta_p)'$ is of the form u_m for some $m \in \mathcal{M}_{\tau,p}(X,E')$. Thus, we have the following

Theorem 2.7 *a) For each $p \in cs(E)$, the map*

$$T_p : \mathcal{M}_{\tau,p}(X,E') \to (C_b(X,E),\beta_p)', m \mapsto u_m,$$

is an algebraic isomorphism.
b) $\mathcal{M}_\tau(X,E')$ is algebraically isomorphic to the dual space of $(C_b(X,E),\beta')$.
c) A subset H of $\mathcal{M}_\tau(X,E')$ is β_p-equicontinuous iff $\sup_{p\in H} m_p(X) < \infty$ and $\sup_{m\in H} m_p(A_\delta) \to 0$ for each net (A_δ) of clopen subsets of X which decreases to the empty set.

We will show now that $\mathcal{M}_{\tau,p}(X,E') = M_{t,p}(X,E')$. We need the following

Lemma 2.8 *Let $\mathcal{F}$ be a family of clopen subsets of X and let $m \in \mathcal{M}_{\tau,p}(X,E')$. If W is a clopen subset of $\bigcup\{A : A \in \mathcal{F}\}$, then there exists $A \in \mathcal{F}$ such that $m_p(W) \leq m_p(A)$.*

Proof: Since, for clopen sets A, B, we have $m_p(A \cup B) = \max\{m_p(A), m_p(B)\}$, we may assume that $\mathcal{F}$ is closed under finite unions. The family $\mathcal{B} = \{W \setminus A : A \in \mathcal{F}\}$ is downwards directed to the empty set. If $m_p(W) > 0$, there exists $A \in \mathcal{F}$ with $m_p(W \setminus A) < m_p(W)$ and so $m_p(W) = \max\{m_p(W \cap A), m_p(W \setminus A)\} = m_p(W \cap A) \leq m_p(A)$, and the result follows.

The proof of the following Theorem is analogous to the proof of Theorem 7.6 in [16].

Theorem 2.9 *Let $m \in \mathcal{M}_{\tau,p}(X,E')$ and let*

$$N_{m,p} : X \to \mathbb{R}, N_{m,p}(x) = \inf\{m_p(A) : x \in A \in K(X)\}.$$

Then: a) $N_{m,p}$ is upper semicontinuous.
b) For each $\epsilon > 0$, the set $X_{m,p,\epsilon} = \{x \in X : N_{m,p}(x) \geq \epsilon\}$ is compact.

Proof: a) It suffices to show that, for each $\theta > 0$, the set $W = \{x \in X : N_{m,p}(x) < \theta\}$ is open. So let $x \in W$. There exists a clopen neighborhood V of x such that $m_p(V) < \theta$. Then $V \subset W$.
b) Let $\mathcal{F}$ be a clopen cover of $X_{m,p,\epsilon}$. Without loss of generality, we may assume that $\mathcal{F}$ is closed under finite unions. The set $M = X \setminus X_{m,p,\epsilon}$ is a union of clopen sets. Thus the family $\mathcal{V} = \{X \setminus (V \cup W) : V \in \mathcal{F}, W \in K(X), W \subset M\}$ is downwards directed to the empty set. Thus, there exists $V \in \mathcal{F}, W \in K(X), W \subset M$ such that $m_p(X \setminus (V \cup W)) < \epsilon$. Hence $X_{m,p,\epsilon} \subset V \cup W$ and so $X_{m,p,\epsilon} \subset V$, which completes the proof.

Theorem 2.10 $\mathcal{M}_{\tau,p}(X,E') = M_{t,p}(X,E')$.

Proof: Let $m \in \mathcal{M}_{\tau,p}(X,E'), \epsilon > 0$ and A a clopen set disjoint from the compact set $Y = X_{m,p,\epsilon}$. Every $x \in A$ has a clopen neighborhood V_x with $m_p(V_x) < \epsilon$. In view of Lemma 2.8, we have that $m_p(A) \leq \epsilon$, which proves that m is tight.

Conversely, assume that m is tight and let (A_δ) be a net of clopen sets decreasing to the empty set. Given $\epsilon > 0$, let Y be a compact subset of X such that $m_p(A) \leq \epsilon$ if the clopen set A is disjoint from Y. There exists some A_δ which is disjoint from Y and so $m_p(A_\delta) \leq \epsilon$. Hence the result follows.

Corollary 2.11 $\mathcal{M}_\tau(X,E') = M_t(X,E')$.

Recall that a subset H of $M(X,E')$ is called tight (see [7], Definition 3.5) if there exists $p \in cs(E)$ such that: (1) $\sup_{m \in H} m_p(X) < \infty$.
(2) For every $\epsilon > 0$, there exists a compact subset Y of X such that $m_p(A) \leq \epsilon$ for every $m \in H$ and every clopen set A disjoint from Y.
By Theorem 3.6 in [7], a subset H of $M_t(X,E')$ is tight iff it is β_o-equicontinuous.

Theorem 2.12 *A subset H of $\mathcal{M}_\tau(X,E')$ is β'-equicontinuous iff it is β_o-equicontinuous.*

Proof: Since β_o is coarser than β', it suffices to show that every β'-equicontinuous subset of $\mathcal{M}_\tau(X, E')$ is β_o-equicontinuous. So let H be such a set. Then H is β_p-equicontinuous for some $p \in cs(E)$. In view of Theorem 2.7, we have that $\sup_{m\in H} m_p(X) < \infty$. Define

$$N_{H,p} : X \to \mathbb{R}, N_{H,p}(x) = \inf\{\sup_{m\in H} m_p(V) : x \in V \in K(X)\}.$$

Using Theorem 2.7, we get (as in the proof of Theorem 2.9) that $N_{H,p}$ is upper-semicontinuous and the set $Y_{H,\epsilon} = \{x \in X : N_{H,p}(x) \geq \epsilon\}$ is compact for every $\epsilon > 0$. For each $V \in K(X)$ disjoint from $Y_{H,\epsilon}$ and each $m \in H$, we have that $m_p(V) \leq \epsilon$. This proves that H is tight and so it is β_o-equicontinuous. Hence the result follows.

Corollary 2.13 *If E is a polar space, then β_o is the polar topology associated with β'.*

Proof: When E is polar, the space $(C_b(X, E), \beta_o)$ is polar. Now the result follows from the preceding Theorem.

Next we will look at the dual space of $C_b(X, E)$ under the topologies β_1 and β_1'. We only sketch the proof of the following Theorem since it is analogous to the corresponding proof given in [16], p. 49, for the case $E = \mathbb{K}$.

Theorem 2.14 *If X is $\mathbb{N}$-replete and E metrizable, then $f(X)$ has non-measurable cardinal for every $f \in C_b(X, E)$.*

Proof: Let d be an ultrametric on E generating its topology. For each positive integer n, consider the equivalence relation $\sim=\sim_n$ on X defined by $x \sim y$ iff $d(f(x), f(y)) \leq 1/n$. Let B_n be a subset of X having only one point in common with each equivalence class. Since X is $\mathbb{N}$-replete, B_n has non-measurable cardinal (see [16], Theorem 2.10). Let $A_n = f(B_n)$. For each $z \in f(X)$ choose a $\bar{z} \in G = \prod A_n$ such that $\bar{z}_n \in A_n$ and $d(z, \bar{z}_n) \leq 1/n$ for each n. In this way we get a map $\phi : f(X) \to G, \phi(z) = \bar{z}$. Since each A_n has non-measurable cardinal, it follows that G has non-measurable cardinal. The result now follows from the fact that ϕ is one-to-one.

Using an argument analogous to the one used in [16], Theorem 7.1, we get the following

Theorem 2.15 *If X is $\mathbb{N}$-replete, then $\mathcal{M}_{\sigma,p}(X, E') = \mathcal{M}_{\tau,p}(X, E')$ for all $p \in cs(E)$.*

Theorem 2.16 *Assume that E is metrizable and let $m \in \mathcal{M}_\sigma(X, E')$. If $f \in C_b(X, E)$ is such that $f(X)$ has non-measurable cardinal, then f is m-integrable.*

Proof: The space $f(X)$ is $\mathbb{N}$-replete since it is ultraparacompact and has non-measurable cardinal (see [16], Theorem 2.18). Hence, there exists a continuous extension f^{v_o} of f to all of v_oX. Let $m^{v_o} : K(v_oX) \to E', m^{v_o}(A) = m(A \cap X)$. Then $m^{v_o} \in \mathcal{M}_{\sigma,p}(v_oX, E')$. Since v_oX is $\mathbb{N}$-replete, we have that $m^{v_o} \in \mathcal{M}_\tau(v_oX, E')$ (in view of the preceding Theorem). Thus f^{v_o} is m^{v_o}-integrable, from which it follows easily that f is m-integrable.

Theorem 2.17 *For every β_1'-continuous linear functional ϕ on $C_b(X, E)$ there exists $m \in \mathcal{M}_\sigma(X, E')$ such that $\phi(f) = \int f dm$ for all $f \in C_{rc}(X, E)$.*

Proof: Since β_1' is coarser than τ_u, there exists (by [6], Theorem 2.8) an $m \in M(X, E')$ such that $\phi(f) = \int f dm$ for all $f \in C_{rc}(X, E)$. Also, there exists $p \in cs(E)$ such that the set $\{f : |\phi(f)| \leq 1\}$ is a $\beta_{1,p}$-neighborhood of zero in $C_b(X, E)$. Let now (A_n) be a sequence of clopen subsets of X, with $A_n \downarrow \emptyset$, and $Q = \bigcap \bar{A}_n^{\beta_o X}$. Let $\lambda \in \mathbb{K}$, with $|\lambda| > 1$, and let μ be a non-zero element of $\mathbb{K}$. There exist a clopen subset B of X, with $\bar{B}^{\beta_o X} \bigcap Q = \emptyset$, and $\epsilon > 0$ such that

$$\{f \in C_b(X, E) : \|f\|_p \leq 1, \|f\|_{B,p} \leq \epsilon\} \subset \{f : |\phi(f)| \leq 1\}.$$

Let n be such that $\bar{B}^{\beta_o X} \bigcap \bar{A}_n^{\beta_o X} = \emptyset$. It follows now easily that $m_p(A_n) \leq |\lambda\mu|$ and hence $m \in \mathcal{M}_{\sigma,p}(X, E')$. Thus the result follows.

Theorem 2.18 *Let E be metrizable and assume that $f(X)$ has non-measurable cardinal for every $f \in C_b(X, E)$. If $m \in \mathcal{M}_\sigma(X, E')$, then the linear functional u_m on $C_b(X, E)$ is β_1'-continuous.*

Proof: Let $m \in \mathcal{M}_{\sigma,p}(X, E')$. Under the hypotheses of the Theorem, every $f \in C_b(X, E)$ is m-integrable. Let $Q \in \Omega_1$. The exists a decreasing sequence (A_n) of clopen subsets of X with $\bigcap \bar{A}_n^{\beta_o X} = Q$. Let $r > 0$ and choose n such that $m_p(A_n) < 1/r$. If B is the complement of A_n in X, then $\bar{B}^{\beta_o X} \cap Q = \emptyset$ and

$$\{f \in C_b(X, E) : \|f\|_p \leq r, \|f\|_{B,p} \leq 1/m_p(X)\} \subset \{f : |u_m(f)| \leq 1\} = W.$$

Thus W is a $\beta_{Q,p}$-neighborhood for every $Q \in \Omega_1$ and so u_m is $\beta_{1,p}$-continuous. This completes the proof.

Theorem 2.19 *If $\phi \in (C_b(X, E), \beta_1)'$, then there exists a unique $m \in M_\sigma(X, E')$ such that $\phi(f) = \int f dm$ for all $f \in C_{rc}(X, E)$.*

Proof: Since β_1 is coarser than τ_u, there exists a unique $m \in M(X, E')$ such that $\phi(f) = \int f dm$ for all $f \in C_{rc}(X, E)$. We neeed to show that $m \in M_\sigma(X, E')$. So, let (A_n) be a sequence of clopen sets which decreases to the empty set. Then $Q = \cap \bar{A}_n^{\beta_o X}$ is in Ω_1. Since ϕ is β_1-continuous, it is $\beta_{Q,p}$-continuous for some $p \in cs(E)$. Let $h \in C_Q$ be such that

$$W_1 = \{f \in C_b(X, E) : \|hf\|_p \leq 1\} \subset \{f : |\phi(f)| \leq 1\}.$$

Taking a $q \geq p$ if necessary, we may assume that $m_p(X) \leq 1$. We will finish the proof by showing that $m_p(A_n) \to 0$. So, let $\lambda \in \mathbb{K}$ with $|\lambda| > 1$ and let μ be a non-zero element of $\mathbb{K}$. There exists n_o such that

$$\overline{A_{n_o}}^{\beta_o X} \subset \{x \in \beta_o X : |h^{\beta_o}(x)| \leq |\mu\lambda^{-1}|\}.$$

It follows easily from this that $m_p(A_{n_o}) \leq |\mu|$ and the result follows.

Theorem 2.20 *Let $m \in M(X, E')$ be such that every $f \in C_b(X, E)$ is m-integrable. Then, u_m is β_1-continuous iff $m \in M_\sigma(X, E')$.*

Poof: The necessity follows from the preceding Theorem. Conversely, assume that $m \in M_\sigma(X, E')$ and let $Q \in \Omega_1$. There exists a decreasing sequence (A_n) of clopen subsets of X

with $Q = \bigcap \overline{A_n}^{\beta_o X}$. Let $p \in cs(E)$ be such that $m_p(X) < \infty$ and $m_p(A_n) \to 0$. Given $r > 0$, choose n such that $m_p(A_n) < 1/r$. If $B = X \setminus A_n$, then $\bar{B}^{\beta_o X}$ is disjoint from Q and

$$\{f \in C_b(X, E) : \|f\|_p \leq r, \|f\|_{B,p} \leq 1/m_p(X)\} \subset \{f : |u_m(f)| \leq 1\}.$$

Thus the set $W = \{f \in C_b(X, E) : |u_m(f)| \leq 1\}$ is a β_Q neighborhood of zero, for each $Q \in \Omega_1$, and so u_m is β_1-continuous, which was to be proved.

If X, Y are Hausdorff zero-dimensional topological spaces, then every continuous function $h : X \to Y$ induces a linear map $T_h : C_b(Y, E) \to C_b(X, E), f \mapsto f \circ h$.

Theorem 2.21 *If $h : X \to Y$ is continuous, then the induced map T_h is $\beta - \beta, \beta_1 - \beta_1, \beta' - \beta'$ and $\beta_1' - \beta_1'$ continuous. In case E is polar, T_h is $\beta_o - \beta_o$ continuous.*

Proof: Let W be a convex β-neighborhood of zero in $C_b(X, E)$ and let $V = T_h^{-1}(W)$. Let $h^{\beta_o} : \beta_o X \to \beta_o Y$ be the continuous extension of h. Given $Q \in \Omega(Y)$, there exists a decreasing net (W_δ) of clopen subsets of $\beta_o Y$ with $\cap W_\delta = Q$. Let $V_\delta = (h^{\beta_o})^{-1}(W_\delta), H = \cap V_\delta$. Then $H \in \Omega(X)$. Since W is a β-neighborhood of zero, it is a $\beta_{H,p}$-neighborhood of zero for some $p \in cs(E)$. Thus, given $r > 0$, there exist a clopen subset A of X, whose closure in $\beta_o X$ is disjoint from H, and $\epsilon > 0$ such that

$$\{f \in C_b(X, E) : \|f\|_p \leq r, \|f\|_{A,p} \leq \epsilon\} \subset W.$$

There exists a δ such that $\bar{A}^{\beta_o X}$ is disjoint from V_δ. The set $B = Y \setminus W_\delta \cap Y$ is clopen in Y with $\bar{B}^{\beta_o Y} \cap Q = \emptyset$. Moreover

$$\{g \in C_b(Y, E) : \|g\|_p \leq r, \|g\|_{B,p} \leq \epsilon\} \subset V.$$

This (by Theorem 2.2) implies that V is a β_Q-neighborhood of zero, which proves that T_h is $\beta - \beta$ continuous. Since an absolutely convex subset of $C_b(X, E)$ is a β'-neighborhood of zero iff it is a β_p-neighgborhood for some $p \in cs(E)$, the proof of the $\beta' - \beta'$ continuity of T_h is analogous. Also the proofs for the cases of the topologies β_1 and β_1' are analogous since a subset of $\beta_o Y$ is a $\mathbb{K}$-zero set iff it is the intersection of a decreasing sequence of clopen subsets of $\beta_o Y$. Finally, if E is polar, then β_o is the polar topology associated with β'. Now the $\beta_o - \beta_o$ continuity of T_h follows from the fact that , if T is a continuous linear map between two locally convex spaces G_1, G_2, then T is also continuous with respect to the corresponding polar topologies on G_1, G_2.

3 Supports of Members of $M(X, E')$

Recall that a subset Y of X is a support set for an $m \in M(X, E')$ if $m(A) = 0$ for each clopen A disjoint from Y. Clearly Y is a support set for m iff $\bar{Y}$ is such a set. For an $m \in M(X, E')$, we define

$$supp(m) = \bigcap\{V \in K(X) : m(A) = 0 \text{ if } A \in K(X), A \cap V = \emptyset\}.$$

If $m \in M_\tau(X, E')$, then for each $s \in E$ the set function $ms : K(X) \to \mathbb{K}, (ms)(A) = m(A)s$ is in $M_\tau(X)$ and hence $supp(m)$ is a suppoprt set for m by Theorem 3.5 in [6].

Let now $m \in M_p(X, E')$. For B a subset of X, we define $m_{*,p}(B) = \sup inf_n m_p(V_n)$, where the supremum is taken over all decreasing sequences (V_n) of clopen sets with $\cap V_n \subset B$.

Theorem 3.1 *Let X be ultraparacompact and $m \in M(X, E')$. Then $m \in \mathcal{M}_{\tau,p}(X, E')$ iff $m_p(X) < \infty$, $supp(m)$ is Lindelöff and $m_{*,p}(X \setminus supp(m)) = 0$.*

Proof: Assume that $m \in \mathcal{M}_{\tau,p}(X, E')$ and let (A_n) be a decreasing sequence of clopen sets with $\cap A_n$ disjoint from $supp(m)$. The family $\mathcal{U} = \{V \in K(X) : m_p(X \setminus V) = 0\}$ is downwards directed and $\bigcap_{V \in \mathcal{U}} V = supp(m)$. Thus the family $\{A_n \cap V : n \in \mathbb{N}, V \in \mathcal{U}\}$ is downwards directed to the empty set. Given $\epsilon > 0$, there exist n and $V \in \mathcal{U}$ such that $m_p(A_n) = m_p(A_n \cap V) < \epsilon$, which proves that $m_{*,p}(X \setminus supp(m)) = 0$. Next, let $\mathcal{F}$ be a clopen cover of $supp(m)$. Since X is ultraparacompact, there exists a clopen partition $(A_i)_{i \in I}$ of X which is a refinement of the open cover $\mathcal{F} \bigcup \{X \setminus supp(m)\}$ of X. Let $I_1 = \{i \in I : A_i \cap supp(m) = \emptyset\}$ and $I_2 = I \setminus I_1$. Then $supp(m) \subset \bigcup_{i \in I_2} A_i$. For each finite subset J of I, let $D_J = \bigcup_{i \notin J} A_i$. Then $D_J \downarrow \emptyset$ and so $m_p(D_{J_o}) < \epsilon$ for some finite subset J_o of I. Clearly $m_p(A_i) < \epsilon$ if $i \notin J_o$. Thus the set $M = \{i \in I : m_p(A_i) \neq 0\}$ is countable, $supp(m) \subset \bigcup_{i \in M} A_i$ and $M \subset I_2$. Since each A_i, for $i \in I_2$, is contained in some member of $\mathcal{F}$, it is clear that $supp(m)$ is covered by a countable subfamily of $\mathcal{F}$ and so $supp(m)$ is Lindelöff.

Conversely, assume that $m_p(X) < \infty$, $supp(m)$ is Lindelöff and $m_{*,p}(X \setminus supp(m)) = 0$. Let (Z_α) be a net of clopen subsets of X decreasing to the empty set. There exists an increasing sequence (α_n) such that $supp(m) \subset \bigcup_{n=1}^{\infty} X \setminus Z_{\alpha_n}$. By our hypothesis, given $\epsilon > 0$, there exists n such that $m_p(Z_{\alpha_n}) < \epsilon$, which clearly completes the proof.

Theorem 3.2 *Let X be ultraparacompact and let $m \in \mathcal{M}_{\sigma,p}(X, E')$. Then $m \in \mathcal{M}_{\tau,p}(X, E')$ iff $supp(m)$ is Lindelöff and $m(A) = 0$ if the clopen set A is disjoint from $supp(m)$.*

Proof: Assume that the condition is satisfied and let (Z_n) be a decreasing sequence of clopen sets such that the set $Z = \cap V_n$ does not meet $supp(m)$. Since X is ultranormal, there exists a clopen set V which contains Z and is disjoint from $supp(m)$. Now $Z_n \bigcap (X \setminus V) \downarrow \emptyset$ and so, given $\epsilon > 0$, there exists n with $m_p(Z_n) = m_p(Z_n \bigcap (X \setminus V)) < \epsilon$. Now the result follows from the preceding Theorem.

Theorem 3.3 *Let X be ultraparacompact and $m \in M(X, E')$. Then $m \in M_\tau(X, E')$ iff $supp(m)$ is Lindelöff and, for each decreasing sequence (A_n) of clopen subsets of X with $\cap A_n$ disjoint from $supp(m)$, there exists $p \in cs(E)$ such that $m_p(A_n) \to 0$.*

Proof: Assume that $m \in M_\tau(X, E')$ and let (A_n) be as in the Theorem. The family $\mathcal{U} = \{V \in K(X) : m(A) = 0 \text{ if } A \cap V = 0\}$ is downwards directed and the family $\{A_n \bigcap (X \setminus V) : n \in \mathbb{N}, V \in \mathcal{U}\}$ is downwards directed to the empty set. Since $m \in M_\tau(X, E')$, there exists $p \in cs(E)$ such that $\lim m_p(A_n \bigcap (X \setminus V)) = 0$. Thus, given $\epsilon > 0$, there exist n and $V \in \mathcal{U}$ such that $m_p(A_n) = m_p(A_n \bigcap (X \setminus V)) < \epsilon$ and so $m_p(A_n) \to 0$. Let now $\mathcal{F}$ be a clopen cover of $supp(m)$ and let $(A_i)_{i \in I}$ be a clopen partition of X which is a refinement of the cover $\mathcal{F} \bigcup \{X \setminus supp(m)\}$. For $J \subset I$ finite, set $D_J = \bigcup_{i \notin J} A_i$. Then $D_J \downarrow \emptyset$. Thus, there exists $q \in cs(E)$ with $\lim m_q(D_J) = 0$. Given $\epsilon > 0$, there exists J_o finite with $m_q(D_{J_o}) < \epsilon$. Thus the set $M = \{i \in I : m_q(A_i) \neq 0\}$ is countable. For each $i \in M$, A_i is contained in some member of $\mathcal{F}$. It follows from this that $supp(m)$ is covered by some countable subfamily of $\mathcal{F}$ and so $supp(m)$ is Lindelöff. Conversely, assume that the condition is satisfied and let (Z_α) be a net of clopen sets which decreases to the empty set. There exists an increasing sequence (α_n) with $supp(m) \subset \bigcup_n X \setminus Z_{\alpha_n}$. There exists $p \in cs(E)$ such that $m_p(Z_{\alpha_n}) \to 0$, which implies that $\lim m_p(Z_\alpha) = 0$. Thus the result follows.

4 The Topologies β_e and β'_e

For d a continuous ultrapseudometric on X, we will denote by X_d the quotient space $X/\sim$, where $\sim$ is the equivalence relation defined by $x \sim y$ iff $d(x,y)=0$. If $\tilde{x}_d$ is the equivalence class of x, then X_d becomes an ultrametric space under the metric $\tilde{d}(\tilde{x}_d, \tilde{y}_d) = d(x,y)$. Let $\pi_d : X \to X_d$ be the quotient map. Since π_d is continuous, we get a linear map $T_d : C_b(X_d, E) \to C_b(X, E), T_d f = f \circ \pi_d$. We define $(C_b(X,E), \beta_e)$ to be the locally convex inductive limit of the spaces $(C_b(X_d, E), \beta)$ with respect to the linear maps T_d, where d ranges over the family of all continuous ultrapseudometrics on X. Also, for $p \in cs(E)$, we define $(C_b(X,E), \beta_{e,p})$ to be the locally convex inductive limit of the spaces $(C_b(X_d, E), \beta_p)$ and $\beta'_e = \bigcup_{p \in cs(E)} \beta_{e,p}$. Note that if $p \leq q$, then $\beta_{e,p} \leq \beta_{e,q}$. It is clear that $\beta'_e \leq \beta_e$.

Theorem 4.1 *Let $h : X \to Y$ be a continuous function, where X, Y are zero-dimensional Hausdorff spaces. Then the induced linear map $S_h : C_b(Y,E) \to C_b(X,E), f \mapsto f \circ h$, is $\beta_u - \beta_u$ and $\beta'_u - \beta'_u$ continuous. Also, for $p \in cs(E)$, S_h is $\beta_p - \beta_p$ and $\beta_{u,p} - \beta_{u,p}$ continuous.*

Proof: Let W be a convex β_u-neighborhood of zero in $C_b(X,E)$. If $Q \in \Omega_u(Y)$, then there exists a clopen partition (A_i) of Y such that $\overline{A_i}^{\beta_o Y}$ is disjoint from Q for each i. If $B_i = h^{-1}(A_i)$, then (B_i) is a clopen partition of X and so the complement H in $\beta_o X$ of the set $\bigcup \overline{B_i}^{\beta_o X}$ is in $\Omega_u(X)$. Let $p \in cs(E)$ be such that W is a $\beta_{H,p}$-neighborhood of zero. Given $r > 0$, there exist $\epsilon > 0$ and a clopen subset B of X, with $\bar{B}^{\beta_o X} \cap H = \emptyset$, such that $\{g \in C_b(X,E) : \|g\|_p \leq r, \|g\|_{B,p} \leq \epsilon\} \subset W$. Since $\bar{B}^{\beta_o X} \subset \bigcup \overline{B_i}^{\beta_o X}$, there exists a finite subset J of I such that $\bar{B}^{\beta_o X} \subset \bigcup_{i \in J} \overline{B_i}^{\beta_o X}$. If $A = \bigcup_{i \in J} A_i$, then $\bar{A}^{\beta_o Y} \cap Q = \emptyset$ and

$$\{f \in C_b(Y,E) : \|f\|_p \leq r, \|f\|_{A,p} \leq \epsilon\} \subset S_h^{-1}(W).$$

This proves that $S_h^{-1}(W)$ is a $\beta_{Q,p}$-neighborhood of zero and the $\beta_u - \beta_u$ continuity of S_h follows. The proof of the $\beta'_u - \beta'_u$ continuity is analogous. The proof of the $\beta_p - \beta_p$ continuity is similar to the proof of the $\beta - \beta$ continuity in Theorem 2.21 while the proof of the $\beta_{u,p} - \beta_{u,p}$ continuity is analogous to the one of the $\beta_u - \beta_u$ continuity. Hence the result follows.

Theorem 4.2 *$\beta_u \leq \beta_e$ and $\beta'_u \leq \beta'_e$*

Proof: For an ultrametrizable space Y we have that $\Omega(Y) = \Omega_u(Y)$ and so $\beta = \beta_u$ and $\beta_p = \beta_{u,p}$ for each $p \in cs(E)$. In view of the preceding Theorem, for each continuous ultrapseudometric d on X, T_d is $\beta - \beta_u$ continuous and so $\beta_u \leq \beta_e$. Also, T_d is $\beta_p - \beta_{u,p}$ continuous, which implies that $\beta'_u \leq \beta'_e$.

Theorem 4.3 *Let $h : X \to Y$ be a continuous function, where X, Y are zero-dimensional Hausdorff spaces. Then, the induced linear map S_h is $\beta_e - \beta_e$ and $\beta'_e - \beta'_e$ continuous.*

Proof: Let d be a continuous ultrapseudometric on Y and define d_1 on $X \times X$ by $d_1(x,y) = d(h(x), h(y))$. Then d_1 is continuous. Let $\psi : X_{d_1} \to Y_d, \tilde{x}_{d_1} \mapsto \tilde{h}(x)_d$. Then ψ is well defined and continuous. Let $S_\psi : C_b(Y_d, E) \to C_b(X_{d_1}, E)$ be the induced linear map. Then $T_{d_1} \circ S_\psi = S_h \circ T_d$. Since T_{d_1} is $\beta - \beta_e$ is continuous and S_ψ $\beta - \beta$ continuous, it follows that $S_h \circ T_d$ is $\beta - \beta_e$ continuous. This clearly proves that S_h is $\beta_e - \beta_e$ continuous. The proof of the $\beta_{e'} - \beta'_e$ continuity of S_h is analogous.

Theorem 4.4 *If E is metrizable, then $\beta_e \le \beta_1, \beta'_e \le \beta'_1$ and $\beta_{e,p} \le \beta_{1,p}$ for each $p \in cs(E)$.*

Proof: Assume that there exists a convex β_e-neighborhood W of zero and a $Q \in \Omega_1$ such that W is not a β_Q-neighborhood of zero. Let (p_n) be an increasing sequence of continuous seminorms on E generating its topology and let $h \in C_{rc}(X)$ be such that $Q = \{x \in \beta_o X : h^{\beta_o}(x) = 0\}$. For each positive integer n, let $A_n = \{x \in X : |h(x)| \ge 1/n\}$. Then $\overline{A_n}^{\beta_o X} = \{x \in \beta_o X : |h^{\beta_o}(x)| \ge 1/n\}$. Since W is not a β_{Q,p_n}-neighborhood of zero, there exists $r_n > 0$ such that, for each clopen B, with $\bar{B}^{\beta_o X}$ disjoint from Q, and each $\epsilon > 0$, there exists $f \in C_b(X,E)$ with $\|f\|_{p_n} \le r_n, \|f\|_{B,p_n} \le \epsilon, f \notin W$. Hence, for each positive integer k, there exists $f_{nk} \in C_b(X,E), f_{nk} \notin W, \|f_{nk}\|_{p_n} \le r_n, \|f_{nk}\|_{A_k,p_n} \le 1/k$. Let $\alpha_{nk} > \max\{\|f_{ij}\|_{p_n} : 1 \le i \le n, 1 \le j \le k\}$ and define

$$d(x,y) = \max\left\{|h(x) - |h(y)|,\ \sup_{n,k} \frac{1}{kn\alpha_{nk}} \left[\max_{1\le i\le n, 1\le j\le k} p_n(f_{ij}(x) - f_{ij}(y))\right]\right\}.$$

Then d is a continuous ultra-pseudometric on X and so $T_d^{-1}(W)$ is a β-neighborhood of zero in $C_b(X_d, E)$. The set $H = \pi_d^{\beta_o}(Q)$ is disjoint from X_d. In fact, assume that $\pi_d^{\beta_o}(x) = \pi_d(a)$ for some $a \in X, x \in Q$. There exists a net (x_δ) in X converging to x and so $\pi_d(x_\delta) \to \pi_d(a)$, i.e. $d(x_\delta, a) \to 0$. Since $h(a) \ne 0$, there exists δ_o such that $d(x_\delta, a) < |h(a)|$ and thus $|h(x_\delta)| = |h(a)|$, which contradicts the fact that $h(x_\delta) \to h^{\beta_o}(x) = 0$. So, H is disjoint from X_d and therefore $T_d^{-1}(W)$ is a β_{H,p_n}-neighborhood of zero for some n. There are an $\epsilon > 0$ and a clopen subset A of X_d, with $\bar{A}^{\beta_o X_d} \cap H = \emptyset$, such that $\{g \in C_b(X_d, E) : \|g\|_{p_n} \le r_n, \|g\|_{A,p_n} \le \epsilon\} \subset T_d^{-1}(W)$. Let $B = \pi_d^{-1}(A)$. Then $\bar{B}^{\beta_o X}$ is disjoint from Q and hence $B = \pi_d^{-1}(A) \subset \bigcup_n \{x : |h^{\beta_o}(x)| \ge 1/n\} = \bigcup_n \overline{A_n}^{\beta_o X}$. Choose $k > 1/\epsilon$ such that $B = \pi_d^{-1}(A) \subset \overline{A_k}^{\beta_o X}$. The function $g : X_d \to E, g(\tilde{x}_d) = f_{nk}(x)$ is well defined, continuous and $T_d g = f_{nk}$. Since $\|g\|_{p_n} \le r_n$ and $\|g\|_{A,p_n} = \|f_{nk}\|_{B,p_n} \le 1/k \le \epsilon$, we have that $g \in T_d^{-1}(W)$ and so $f_{nk} \in W$, a contradiction. This proves that $\beta_e \le \beta_1$. Suppose next that there exists a convex $\beta_{e,p}$-neighborhood W of zero which is not a $\beta_{Q,p}$-neighborhood for some $Q \in \Omega_1$. There exists $r > 0$ such that, for every clopen subset B of X , whose closure in $\beta_o X$ is disjoint from Q, and any $\epsilon > 0$, there exists $f \in C_b(X,E)$ with $\|f\|_p \le r, \|f\|_{B,p} \le \epsilon, f \notin W$. Let $h \in C_{rc}(X)$ be such that $Q = \{x \in \beta_o X : h^{\beta_o}(x) = 0\}$ and let $A_n = \{x : |h(x)| \ge 1/n\}$. For each n, there exists an $f_n \in C_b(X,E)$ with $\|f_n\|_p \le r, \|f_n\|_{A_n,p} \le 1/n, f_n \notin W$. Let (p_n) be an increasing sequence of continuous seminorms on E generating its topology. Choose $\alpha_n > \max\{\|f_k\|_{p_n} : 1 \le k \le n\}$ and define

$$d(x,y) = \max\left\{|h(x) - h(y)|, \sup_n \frac{1}{n\alpha_n}\left[\max_{1\le k\le n} p_n(f_k(x) - f_k(y))\right]\right\}.$$

Then d is a continuous ultra-pseudometric on X and so $T_d^{-1}(W)$ is a β_p-neighborhood of zero. Since $H = \pi_d^{\beta_o}(Q)$ is disjoint from X_d, there exist a clopen subset A of X_d, with $\bar{A}^{\beta_o X_d} \cap H = \emptyset$, and $\epsilon > 0$ such that

$$\{g \in C_b(X_d, E) : \|g\|_p \le r, \|g\|_{A,p} \le \epsilon\} \subset T_d^{-1}(W).$$

Choose $n > 1/\epsilon$ such that $\bar{B}^{\beta_o X} \subset \overline{A_n}^{\beta_o X}$, where $B = \pi_d^{-1}(A)$. The function $g : X_d \to E, g(\tilde{x}_d) = f_n(x)$, is well defined and continuous. Since $\|g\|_p = \|f_n\|_p \le r$ and $\|g\|_{A,p} = \|f_n\|_{B,p} \le 1/n \le \epsilon$ we have that $f_n = T_d g \in W$, a contradiction. This proves that $\beta_{e,p} \le \beta_{1,p}$, for each $p \in cs(E)$, which implies that $\beta'_e \le \beta'_1$. This completes the proof.

Theorem 4.5 *Assume that β_e is coarser than τ_u, e.g. when E is metrizable. Then, on each uniformly bounded equicontinuous subset H of $C_b(X,E)$, β_e coincides with the topology τ_s of pointwise convergence.*

Proof: We may assume that H is absolutely convex. Let W be a convex β_e-neighborhood of zero. Since β_e is coarser than τ_u, there exists $p \in cs(E)$ such that $W_1 = \{f \in C_b(X,E) : \|f\|_p \le 1\} \subset W$. Consider the continuous ultra-pseudometric d on X defined by $d(x,y) = \sup_{f\in H} p(f(x) - f(y))$ and let $(A_i)_{i\in I}$ be the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $d(x,y) \le 1$. Let $V = T_d^{-1}(W)$ and $Q = \beta_o X \setminus \bigcup_{i\in I} \overline{A_i}^{\beta_o X}$. Then $D = \pi_d^{\beta_o}(Q)$ is disjoint from X_d. Let $q \in cs(E), q \ge p$, be such that V is a $\beta_{D,q}$-neighborhood of zero. If $r > \sup_{f\in H} \|f\|_q$, then there exist $\epsilon_1 > 0$ and a clopen set B in X_d, with $\bar{B}^{\beta_o X_d} \cap D = \emptyset$, such that $\{g \in C_b(X_d,E) : \|g\|_q \le r, \|g\|_{D,q} \le \epsilon_1\} \subset V$. If $A = \pi_d^{-1}(B)$, then $\bar{A}^{\beta_o X} \subset \bigcup \bar{A_i}^{\beta_o X}$ and so $\bar{A}^{\beta_o X} \subset \bigcup_{i\in J} \bar{A_i}^{\beta_o X}$ for some finite subset J of I. Choose $x_i \in A_i$, for each $i \in I$, and let $W_2 = \{f \in C_b(X,E) : p(f(x_i)) \le \epsilon_1 \quad \text{for} \quad i \in J\}$. Then $W_2 \cap H \subset W$. Indeed, let $f \in W_2 \cap H$ and let $f^* = \sum_{i\in I} \chi_{A_i} f(x_i)$. The function $h : X_d \to E, h(\tilde{x}_d) = f^*(x)$ is well defined, bounded, continuous and $T_d h = f^*$. Moreover, $\|f^*\|_q \le r$. If $\tilde{x}_d \in B$, then $x \in A$ and so $x \in A_i$, for some $i \in J$, which implies that $h(\tilde{x}_d) = f(x_i)$. Thus $\|h\|_{B,q} \le \epsilon_1$ and therefore $h \in V$, i.e $f^* \in W$. Also $\|f - f^*\|_p \le 1$ and so $f - f^* \in W$. Thus $f \in W$. This proves that the topology induced on H by τ_s is finer than the one induced by β_e and the proof is complete since τ_s is coarser that β_e.

The proof of the following Theorem is analogous to the one of the preceding Theorem.

Theorem 4.6 *For $p \in cs(E)$, let τ_p be the topology on $C_b(X,E)$ generated by the seminorm $\|.\|_p$. If $\beta_{e,p}$ is coarser than τ_p, then on τ_p-bounded p-equicontinuous subsets of $C_b(X,E)$, $\beta_{e,p}$ coincides with the topology generated by the seminorms $f \mapsto p(f(x)), x \in X$.*

Theorem 4.7 *Assume that τ_u is finer than β_e and let W be a convex β_e-neighborhood of zero. Then, for each $f \in C_b(X,E)$, there are pairwise disjoint clopen sets $A_1, \dots A_n$ in X and $x_k \in A_k$ such that $f - \sum_{k=1}^n \chi_{A_k} f(x_k) \in W$.*

Proof: We may assume that W is convex. Since τ_u is finer than β_e , there exists $p \in cs(E)$ such that $W_1 = \{g \in C_b(X,E) : \|g\|_p \le 1\} \subset W$. Let $(A_i)_{i\in I}$ be the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $p(f(x) - f(y)) \le 1$. Let $h_i = \chi_{A_i}, x_i \in A_i, f^* = \sum_{i\in I} h_i f(x_i)$. Then $f - f^* \in W_1$. The ultra-pseudometric $d(x,y) = \sup_i |h_i(x) - h_i(y)|$ is continuous and so $V = T_d^{-1}(W)$ is a β-neighborhood of zero in $C_b(X_d, E)$. Let $Q = \beta_o X \setminus \bigcup_i \overline{A_i}^{\beta_o X}$ and $D = \pi_d^{\beta_o}(Q)$. Then D is disjoint from X_d and hence there exists $q \in cs(E)$ such that V is a $\beta_{D,q}$-neighborhood of zero. There are $\epsilon > 0$ and a clopen subset A of X_d, with $\bar{A}^{\beta_o X_d} \cap D = \emptyset$, such that

$$\{g \in C_b(X_d,E) : \|g\|_q \le \|f\|_q, \|g\|_{A,q} \le \epsilon\} \subset V.$$

If $B = \pi_d^{-1}(A)$, then $\bar{B}^{\beta_o X} \subset \bigcup_i \overline{A_i}^{\beta_o X}$ and so $\bar{B}^{\beta_o X} \subset \bigcup_{i\in J} \overline{A_i}^{\beta_o X}$ for some finite subset J of I. Let $g = \sum_{i\in J} h_i f(x_i), g_1 = f^* - g$. The function $\bar{g_1} : X_d \to E, \bar{g_1}(\tilde{x}_d) = g_1(x)$ is well defined, continuous and $T_d \bar{g_1} = g_1$. Since $\|\bar{g_1}\|_q \le \|f\|_q$ and $\bar{g_1} = 0$ on A, it follows that $\bar{g_1} \in V$ and so $g_1 \in W$. Finally, $f - g = (f - f^*) + g_1 \in W$, which completes the proof.

5 The Dual Spaces of $(C_b(X,E),\beta_u)$ and $(C_b(X,E),\beta'_u)$

We will denote by $M_u(X,E')$ the space of all $m \in M(X,E')$ with the following property: For each decreasing net (A_δ) of clopen subsets of X with $\bigcap_\delta \overline{A_\delta}^{\beta_o X} \in \Omega_u$, there exists $p \in cs(E)$ such that $m_p(X) < \infty$ and $m_p(A_\delta) \to 0$.

Theorem 5.1 *If $m \in M_u(X,E')$, then every $f \in C_b(X,E)$ is m-integrable and the linear functional $u_m(f) = \int f dm, f \in C_b(X,E)$, is β_u-continuous.*

Proof: There exists a $p \in cs(E)$ such that $m_p(X) \leq 1$. Let $f \in C_b(X,E)$ and $\epsilon > 0$. Without loss of generality, we may assume that $\|f\|_p \leq 1$. Let $(A_i)_{i\in I}$ be the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $p(f(x) - f(y)) \leq \epsilon$. For $J \subset I$ finite, set $B_J = \bigcup_{i\notin J} A_i$. Then (B_J) is a decreasing net of clopen sets with $Q = \bigcap \overline{B_J}^{\beta_o X} = \beta_o X \setminus \bigcup_{i\in I} \overline{A_i}^{\beta_o X}$ and hence $Q \in \Omega_u$. Thus, there exists $q \in cs(E), q \geq p$, such that $m_q(B_J) \to 0$. We now finish the proof by using an argument analogous to the one used in the proof of Theorem 2.4.

Argueing as in the proof of Theorem 2.5, we get the following

Theorem 5.2 *The map $m \mapsto u_m$, from $M_u(X,E')$ to $(C_b(X,E),\beta_u)'$, is an algebraic isomorphism.*

Next we will look at the dual space of $(C_b(X,E),\beta'_u)$. For $p \in cs(E)$, let $\mathcal{M}_{u,p}(X.E')$ be the space of all $m \in M_p(X,E')$ with the following property: For each decreasing net (A_δ) of clopen subsets of X, with $\bigcap_\delta \overline{A_\delta}^{\beta_o X} \in \Omega_u$, we have that $m_p(A_\delta) \to 0$. Let

$$\mathcal{M}_u(X,E') = \bigcup_{p\in cs(E)} \mathcal{M}_{u,p}(X.E').$$

Clearly $\mathcal{M}_u(X,E' \subset M_u(X,E')$. Using arguments analogous to the ones used in the proofs of Theorems 2.4 and 2.5, we get the following

Theorem 5.3 *a) For each $m \in \mathcal{M}_{u,p}(X.E'), u_m$ is $\beta_{u,p}$-continuous and the map $m \mapsto u_m$, from $\mathcal{M}_{u,p}(X,E')$ to $(C_b(X,E),\beta_{u,p})'$, is an algebraic isomorphism.*
b) $\mathcal{M}_u(X,E')$ is algebraically isomorphic to $(C_b(X,E),\beta'_u)'$ via the isomorphism $m \mapsto u_m$.

Theorem 5.4 *Let $m \in M(X,E')$. Then 1) $m \in M_u(X,E')$ iff the following condition is satisfied: For each clopen partition $(A_i)_{i\in I}$ of X, there exists $p \in cs(E)$, with $m_p(X) < \infty$, such that, for each $\epsilon > 0$, there exists a finite subset J of I with $m_p(\bigcup_{i\notin J} A_i) \leq \epsilon$.*
2) $m \in \mathcal{M}_{u,p}(X,E')$ iff $m_p(X) < \infty$ and, for each clopen partition $(A_i)_{i\in I}$ of X and each $\epsilon > 0$, there exists a finite subset J of I such that $m_p(\bigcup_{i\notin J} A_i) \leq \epsilon$.

Proof: 1) Assume that $m \in M_u(X,E')$ and let $(A_i)_{i\in I}$ be a clopen partition of X. For $J \subset I$ finite, set $B_J = \bigcup_{i\notin J} A_i$. Then $Q = \bigcap \overline{B_J}^{\beta_o X} \in \Omega_u$ and hence there exists $p \in cs(E)$, with $m_p(X) < \infty$, such that $m_p(B_J) \to 0$.
Conversely, assume that the condition is satisfied and let (B_δ) be a decreasing net of clopen sets with $\bigcap \overline{B_\delta}^{\beta_o X} = D \in \Omega_u$. There exists a clopen partition $(A_i)_{i\in I}$ of X such that each $\overline{A_i}^{\beta_o X}$ is disjoint from D. Let $p \in cs(E)$ be as in the condition. Given $\epsilon > 0$, there

exists $J \subset I$ finite such that $m_p(D_J) \leq \epsilon$, where $D_J = \bigcup_{i \notin J} A_i$. If $M_J = X \setminus D_J$, then $\overline{M_J}^{\beta_o X} = \bigcup_{i\in J} \overline{A_i}^{\beta_o X} \subset \bigcup_\delta \beta_o X \setminus \overline{B_\delta}^{\beta_o X}$. There exists δ such that $\overline{M_J}^{\beta_o X} \subset \beta_o X \setminus \overline{B_\delta}^{\beta_o X}$. Thus $\overline{B_\delta}^{\beta_o X} \subset \overline{D_J}^{\beta_o X}$ and so $m_p(B_\delta) \leq \epsilon$, which proves 1).
2) The proof is analogous to that of 1).

Remark 5.5 *In view of the preceding Theorem, $\mathcal{M}_u(X, E')$ coincides with the space $M'_u(X, E')$ introduced in [3]. In the same paper it was shown that the dual space of $(C_{rc}(X, E), \beta'_u)$ is $M'_u(X, E')$.*

Theorem 5.6 *For a subset H of $M_u(X, E')$, the following are equivalent:*
(1) H is β_u-equicontinuous.
(2) For each decreasing net (A_δ) of clopen subsets of X with $\bigcap_\delta \overline{A_\delta}^{\beta_o X} \in \Omega_u$, there exists $p \in cs(E)$ such that $\sup_{m\in H} m_p(X) < \infty$ and $m_p(A_\delta) \to 0$ uniformly for $m \in H$.
(3) For each clopen partition $(A_i)_{i\in I}$ of X, there exists $p \in cs(E)$, with $\sup_{m\in H} m_p(X) < \infty$, such that, for each $\epsilon > 0$, there exists a finite subset J of I with $m_p(\bigcup_{i\notin J} A_i) \leq \epsilon$ for all $m \in H$.

Proof: The equivalence of (2) and (3) can be proved using an argument analogous to the one used in the proof of Theorem 5.4.
(1)$\Rightarrow$ (2). Let H^o be the polar of H in $C_b(X, E)$. Since $\beta_u \leq \tau_u$, there exists $q \in cs(E)$ such that $\{f \in C_b(X, E) : \|f\|_q \leq 1\} \subset H^o$. It follows from this that $\sup_{m\in H} m_q(X) < \infty$. Let now (A_δ) be a decreasing net of clopen sets with $Q = \bigcap_\delta \overline{A_\delta}^{\beta_o X} \in \Omega_u$. Since H^o is a β_u-neighborhood of zero, there exist $p \in cs(E), p \geq q$, and $h \in C_Q$ such that $W_2 = \{f \in C_b(X, E) : \|hf\|_p \leq 1\} \subset H^o$. We will prove that $\lim_\delta m_p(A_\delta) = 0$ uniformly for $m \in H$. So let μ be a non-zero element of $\mathbb{K}$. The set $D = \{x \in \beta_o X : |h^{\beta_o}(x)| \leq |\mu|\}$ contains Q and so it contains some $\bar{A}_\delta^{\beta_o X}$. If now A is a clopen subset of X contained in A_δ and $s \in E$ with $p(s) \leq 1$, then $\mu^{-1}\chi_A s \in W_2$ and thus $|m(A)s| \leq |\mu|$ for all $m \in H$. If $\lambda \in \mathbb{K}, |\lambda| > 1$, then $m_p(A_\delta) \leq |\mu\lambda|$, which proves that $m_p(A_\delta) \to 0$ uniformly for $m \in H$.
(2)$\Rightarrow$ (1). Let $Q \in \Omega_u$. There exists a decreasing net (A_δ) of clopen subsets of X such that $\bigcap \bar{A}_\delta^{\beta_o X} = Q$. Let p be as in (2) and let $d > \sup_{m\in H} m_p(X)$. Given $r > 0$, there exists a δ such that $m_p(A_\delta) < 1/r$ for all $m \in H$. If $B = X \setminus A_\delta$, then $\bar{B}^{\beta_o X}$ is disjoint from Q and $\{f \in C_b(X, E) : \|f\|_p \leq r, \|f\|_{B,p} \leq 1/d\} \subset H^o$, which proves that H^o is a $\beta_{Q,p}$-neighborhood of zero. It follows that H^o is a β_u-neighborhood and so H is β_u-equicontinuous.

The proof of the following Theorem is analogous to the one of the preceding Theorem.

Theorem 5.7 *For a subset H of $\mathcal{M}_u(X, E')$ and $p \in cs(E)$ the following are equivalent:*
(1) H is $\beta_{u,p}$-equicontinuous.
(2) $\sup_{m\in H} m_p(X) < \infty$ and for each decreasing net (A_δ) of clopen subsets of X with $\bigcap_\delta \overline{A_\delta}^{\beta_o X} \in \Omega_u$, we have that $m_p(A_\delta) \to 0$ uniformly for $m \in H$.
(3) $\sup_{m\in H} m_p(X) < \infty$ and for each clopen partition $(A_i)_{i\in I}$ of X and each $\epsilon > 0$, there exists a finite subset J of I such that $m_p(\bigcup_{i\notin J} A_i) \leq \epsilon$ for each $m \in H$.

6 The Dual Spaces of $(C_b(X,E),\beta_e)$ and $(C_b(X,E),\beta'_e)$

Theorem 6.1 *Assume that β_e is coarser than τ_u and let u be a β_e-continuous linear form on $C_b(X,E)$. Let $m \in M(X,E')$ be such that $u(f) = \int f dm$ for all $f \in C_{rc}(X,E)$. For $f \in C_b(X,E)$ and $A \in K(X)$, set $|m|_f(A) = \sup\{|m(B)f(x)| : x \in X, B \in K(X), B \subset A\}$. If $(A_i)_{i\in I}$ is a clopen partition of X, then:*
(1) For each $g \in C_b(X,E)$ of the form $g = \sum \chi_{A_i}s_i$, $s_i \in E$, we have that $u(g) = \sum_i m(A_i)s_i$.
(2) For each $\epsilon > 0$, the set $I_\epsilon = \{i \in I : |m|_f(A_i) \geq \epsilon\}$ is finite.
(3) If $x_i \in A_i$, then the function $f^* = \sum_i \chi_{A_i}f(x_i)$ is m-integrable.

Proof: (1) For $J \subset I$ finite, let $h_J = \sum_{i\in J}\chi_{A_i}s_i$. The set $\{h_J : J \text{ finite}\}$ is uniformly bounded, equicontinuous and $h_J \to g$ pointwise. By Theorem 4.5, we have that $\sum_{i\in J} m(A_i)s_i = u(h_J) \to u(g)$.
(2) For each i, there exist a clopen subset B_i of A_i and $x_i \in X$ such that $|m(B_i)f(x_i)| \geq |m|_f(A_i)/2$. The set $B = \cup_i B_i$ is clopen. Using (1), we get that $u(\sum_{i\in I}\chi_{B_i}f(x_i)) = \sum_{i\in I} m(B_i)f(x_i)$. There exists a finite subset J of I such that $|m(B_i)f(x_i)| < \epsilon/2$ if $i \notin J$. For such i, we have $|m|_f(A_i) < \epsilon$ and so $i \notin I_\epsilon$.
(3) Let $\epsilon > 0$ and $D = \bigcup_{i\notin I_\epsilon} A_i$. Let $\{D_1, \dots, D_N\}$ be a clopen partition of X, which is a refinement of $\{A_i : i \in I_\epsilon\} \cup \{D\}$, and let $y_k \in D_k$. We may assume that $\bigcup_{k=1}^r D_k = \bigcup_{i\in I_\epsilon} A_i$. Then $\sum_{k=1}^r m(D_k)f^*(y_k) = \sum_{i\in I_\epsilon} m(A_i)f(x_i)$. Let A be a clopen subset of D and $z \in A$. Using (1), we get that $m(A)f^*(z) = \sum_{i\notin I_\epsilon} m(A\cap A_i)f^*(z)$. But, for $i \notin I_\epsilon$, $|m(A\cap A_i)f^*(z)| \leq |m|_f(A_i) < \epsilon$. Thus $|m(A)f^*(z)| \leq \epsilon$ and so $|\sum_{k=r+1}^N m(D_k)f^*(y_k)| \leq \epsilon$, which implies that $|\sum_{k=1}^N m(D_k)f^*(y_k) - \sum_{i\in I_\epsilon} m(A_i)f(x_i)| \leq \epsilon$. It clearly follows that f^* is m-integrable.

Theorem 6.2 *Assume that β_e is coarser than τ_u and let u and m be as in the preceding Theorem. Then: (1) Every $f \in C_b(X,E)$ is m-integrable.*
(2) If $(A_i)_{i\in I}$ is a clopen partition of X and $(s_i)_{i\in I}$ a bounded family in $\mathbb{K}$, then for $g = \sum \chi_{A_i}s_i$, we have that $\int g dm = u(g) = \sum_i m(A_i)s_i$.
(3) $u(f) = \int f dm$ for each $f \in C_b(X,E)$.

Proof: (1) Let $\epsilon > 0$ and let $p \in cs(E)$ be such that $m_p(X) \leq 1$. Let $(A_i)_{i\in I}$ be the clopen partition of X which corresponds to the equivalence relation $x \sim y$ iff $p(f(x) - f(y)) \leq \epsilon$. Let $x_i \in A_i$, $f^* = \sum \chi_{A_i}f(x_i)$. Since f^* is m-integrable, there exists a clopen partition $\{Z_1, \dots, Z_n\}$ of X and $z_k \in Z_k$ such that, for each clopen partition $\{D_1, \dots, D_N\}$ of X, which is a refinement of $\{Z_1, \dots, Z_n\}$, and any $y_j \in D_j$, we have $|\sum_{k=1}^n m(Z_k)f^*(z_k) - \sum_{j=1}^N m(D_j)f^*(y_j)| \leq \epsilon$. Since, for $h = f - f^*$, we have $|m(A)h(z)| \leq \epsilon$ for all $A \in K(X)$ and all $z \in X$, we have that $|\sum_{j=1}^N m(D_j)f(y_j) - \sum_{k=1}^n m(Z_k)f(z_k)| \leq \epsilon$, which proves that f is m-integrable.
(2) Given $\epsilon > 0$, there exists a finite subset J_o of I such that $|m|_g(A_i) < \epsilon$ if $i \notin J_o$. If the clopen set A is disjoint from $D = \bigcup_{i\in J_o} A_i$, then $|m(A)g(x)| \leq \epsilon$ for all $x \in X$. Also there exists a finite subset J of I containing J_o such that $|u(g) - \sum_{i\in J} m(A_i)s_i| < \epsilon$. If $h = \sum_{i\notin J}\chi_{A_i}s_i$, then $|\int h dm| \leq \epsilon$ and so $|\int g dm - \sum_{i\in J} m(A_i)s_i| \leq \epsilon$, which implies that $|\int g dm - u(g)| \leq \epsilon$.
(3) Let $f \in C_b(X,E)$ and $\epsilon > 0$. There exists $p \in cs(E)$ such that $m_p(X) \leq 1$ and $W_1 = \{g \in C_b(X,E) : \|g\|_p \leq 1\} \subset \{g : |u(g)| \leq 1\}$. Let $(A_i)_{i\in I}$ be the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $p(f(x) - f(y)) \leq \epsilon$. Choose $x_i \in A_i$ and let $f^* = \sum \chi_{A_i}f(x_i), h = f - f^*$. There exists a finite subset J of I such that

$|u(f^*)-\sum_{i\in J} m(A_i)f(x_i)| < \epsilon$ and $|m|_f(A_i) < \epsilon$ if $i \notin J$. If $D = \bigcup_{i\in J} A_i$ and $Z = X\setminus D$, then $|\int_Z f^*dm| \le \epsilon$ and $\int_D f^*dm = \sum_{i\in J} m(A_i)f(x_i)$ and so $|\int f^*dm - \sum_{i\in J} m(A_i)f(x_i)| \le \epsilon$. Thus $|\int f^*dm - u(f^*)| \le \epsilon$. If $|\lambda| > 1$, then $|u(h)| \le \epsilon|\lambda|$ and $|\int hdm| \le \epsilon$. It follows that $|\int fdm - u(f)| \le \epsilon|\lambda|$, which clearly completes the proof.

By the above Theorem, we have the following

Theorem 6.3 *If β_e is coarser than τ_u, then there exists a subspace $M_e(X,E')$ of $M(X,E')$ such that every $f \in C_b(X,E)$ is m-integrable for every $m \in M_e(X,E')$ and the map $m \mapsto u_m$, from $M_e(X,E')$ to $(C_b(X,E),\beta_e)'$, is an algebraic isomorphism.*

Conjecture 6.4 *If β_e is coarser than τ_u, then $M_e(X,E') = M_u(X,E')$.*

For $p \in cs(E)$, d a continuous ultra-pseudometric on X and A a d-clopen subset of X, we define

$$|m|_{d,p}(A) = \sup\left\{\frac{|m(B)s|}{p(s)} : p(s) \neq 0, B \subset A, B \;\; d-\text{clopen}\right\}.$$

Also, for $Y \subset X$, we define $|m|^*_{d,p}(Y)$ to be the infimum of all $\sup_n m_{d,p}(A_n)$, where the infimum is taken over all sequences (A_n) of d-clopen sets which cover Y. We will need the following

Theorem 6.5 *Let (Z,d) be an ultrametric space and assume that E has the countable neighborhood property. If $m \in M_\tau(Z,E')$, then there exists $p \in cs(E)$, with $m_p(Z) < \infty$, and a d-closed, d-separable subset G of Z such that $|m|^*_{d,p}(Z\setminus G) = 0$.*

Proof: For $Y \subset Z$ finite and $\epsilon > 0$, set $N(Y,\epsilon) = \{x \in Z : d(x,Y) \le \epsilon\}$. The family $\{Z\setminus N(Y,\epsilon) : Y \text{ finite}\}$ is downwards directed to the empty set. Since $m \in M_\tau(Z,E')$, there exists $q \in cs(E)$, with $m_q(Z) < \infty$, such that $\lim_Y |m|_{d,q}(Z\setminus N(Y,\epsilon) = 0$. Hence, there exists an increasing sequence (p_n) in $cs(E)$ such that $\lim_Y |m|_{d,p_n}(Z\setminus N(Y,1/n) = 0$ for all n. Since E has the countable neighborhood property, there exist $p \in cs(E)$ and a sequence (μ_n) of non-zero elements of $\mathbb{K}$ such that $p \ge |\mu_n|p_n$ for all n. For each k, choose an increasing sequence $(Y_{k,n})_n$ of finite subsets of Z such that $|m|_{d,p_k}(Z\setminus N(Y_{k,n},1/k)) < |\mu_k|/n$ for all n. Let $D_n = \bigcup_k (Z\setminus N(Y_{k,n},1/k))$, $M = \bigcup_n Z\setminus D_n$ and $G = \bar{M}$. We have

$$|m|_{d,p}(Z\setminus N(Y_{k,n},1/k)) \le |\mu_k|^{-1}|m|_{d,p_k}(Z\setminus N(Y_{k,n},1/k)) \le 1/n.$$

Thus $|m|^*_{d,p}(Z\setminus G) = 0$. Also, G is d-separable. Indeed, let $x \in G$ and $\epsilon > 0$. There exists $y \in M$ with $d(x,y) < \epsilon$. Let n be such that $y \notin D_n$. Choose $k > 1/\epsilon$. Since $y \in N(Y_{k,n},1/k)$, there exists $z \in Y_{k,n}$ with $d(z,y) \le 1/k < \epsilon$ and so $d(x,z) < \epsilon$. It follows that G is contained in $\bar{L}$, where $L = \bigcup_{n,k} Y_{k,n}$. Since $\bar{L}$ is separable, its subspace G is also separable. This completes the proof.

Let $M_s(X,E')$ be the space of all $m \in M(X,E')$ with the following property: For each continuous ultra-pseudometric d on X, there exist $p \in cs(E)$, with $m_p(X) < \infty$, and a d-closed, d-separable subset G of X such that $|m|^*_{d,p}(Z\setminus G) = 0$.

Theorem 6.6 *If β_e is coarser than τ_u and E has the countable neighborhhod property, then every $m \in M_e(X,E')$ is in $M_s(X,E')$.*

Proof: Let d be a given continuous ultra-pseudometric on X. Since u_m is β_e-continuous, $T_d^* u_m$ is β-continuous on $C_b(X_d, E)$. Thus, there is $\mu \in M_\tau(X_d, E')$ such that $\int g d\mu = \int (T_d g) dm$ for all $g \in C_b(X_d, E)$. By the preceding Theorem, there exists a $\tilde{d}$-closed, $\tilde{d}$-separable subset Z of X_d such that $|\mu|^*_{\tilde{d},p}(X_d \setminus Z) = 0$. The set $G = \pi_d^{-1}(Z)$ is d-closed, d-separable and $|m|^*_{d,p}(X \setminus G) = 0$.

We will look next at the dual space of $(C_b(X, E), \beta_e')$. For $p \in cs(E)$, we denote by $\mathcal{M}_{s,p}(X, E')$ the space of all $m \in \mathcal{M}_{\sigma,p}(X, E')$ with the following property: For each continuous ultra-pseudometric d on X, there exists a d-closed, d-separable subset G of X such that $|m|^*_{d,p}(X \setminus G) = 0$. Let $\mathcal{M}_s(X, E') = \bigcup \mathcal{M}_{s,p}(X, E')$. For the proof of the following theorem we may use an argument analogous to the one used in the proof of Theorem 6.5. Note that in the next Theorem we don't need to assume that E has the countable neighborhood property.

Theorem 6.7 *Let (Z, d) be an ultrametric space and let $p \in cs(E)$ and $m \in \mathcal{M}_{\tau,p}(Z, E')$. Then there exists a d-closed, d-separable subset G of Z such that $|m|^*_{d,p}(Z \setminus G) = 0$*

Theorem 6.8 *If E is metrizable, then $\mathcal{M}_{s,p}(X, E')$ is algebraically isomorphic to the dual space of $(C_b(X, E), \beta_{e,p})$.*

Proof: Let u be in the dual space of $(C_b(X, E), \beta_{e,p})$. By Theorem 6.3, there exists $m \in M_e(X, E')$ such that $u(f) = \int f dm$ for all $f \in C_b(X, E)$. Since $\beta_{e,p} \leq \beta_{1,p}$, m is in $\mathcal{M}_{\sigma,p}(X, E')$ (see Theorem 2.17). We will show that $m \in \mathcal{M}_{s,p}(X, E')$. So, let d be a continuous ultra-pseudometric on X. Since T_d is $\beta_p - \beta_{e,p}$ continuous, $T_d^* u$ is β_p-continuous on $C_b(X_d, E)$ and so there exists $\mu \in \mathcal{M}_{\tau,p}(X_d, E')$ such that $\int g d\mu =< T_d g, u >$ for all $g \in C_b(X_d, E)$. In view of the preceding Theorem, there exists a $\tilde{d}$-closed, $\tilde{d}$-separable subset Z of X_d with $|\mu|^*_{\tilde{d},p}(X_d \setminus Z) = 0$. The set $G = \pi_d^{-1}(Z)$ is d-closed, d-separable and $|m|^*_{d,p}(X \setminus G) = 0$, which proves that $m \in \mathcal{M}_{s,p}(X, E')$. Conversely, let $m \in \mathcal{M}_{s,p}(X, E')$ and let d be a continuous ultra-pseudometric on X. Define $\mu = \mu_d : K(X_d) \to E', \mu(A) = m(\pi_d^{-1}(A))$. Then $\mu \in \mathcal{M}_{\sigma,p}(X_d, E')$ since $m \in \mathcal{M}_{\sigma,p}(X, E')$.
Claim I: $\mu \in \mathcal{M}_{\tau,p}(X_d, E')$. Indeed, let (V_δ) be a net of clopen subsets of X_d which decreases to the empty set. By our hypothesis, there is a d-closed, d-separable subset G of X such that $|m|^*_{d,p}(Z \setminus G) = 0$. Given $\epsilon > 0$, there is an increasing sequence (Z_n) of d-clopen subsets of X covering $X \setminus G$ and such that $|m|_{d,p}(Z_n) < \epsilon$ for all n. If $M = \pi_d(G)$ and $A_n = \pi_d(Z_n)$, then M is closed and separable in X_d and $X_d \setminus M \subset \bigcup A_n$. Moreover, each A_n is clopen in X_d and $|\mu|_{\tilde{d},p}(A_n) \leq \epsilon$. Since M is separable, there exists an increasing sequence (δ_n) such that $M \subset \bigcup_{n=1}^{\infty}(X_d \setminus V_{\delta_n})$. Now $V_{\delta_n} \bigcap (X_d \setminus A_n) \downarrow \emptyset$ and hence there exists n with $|\mu|_{\tilde{d},p}(V_{\delta_n} \cap (X_d \setminus A_n)) \leq \epsilon$, which, together with the $|\mu|_{\tilde{d},p}(A_n) \leq \epsilon$, implies that $|\mu|_{\tilde{d},p}(V_{\delta_n}) \leq \epsilon$. This proves our claim.
Let now d, d_1 be continuous ultra-pseudometrics on X, with $d \leq d_1$, and let $f \in C_b(X, E)$ be d-continuous. The functions $h, h_1 : X_d \to E, h(\tilde{x}_d) = f(x) = h_1(\tilde{x})$ are well defined and continuous. Morerover $\int h d\mu_d = \int h_1 d\mu_{d_1}$. Indeed, let $\phi : X_{d_1} \to X_d, \tilde{x}_{d_1} \mapsto \tilde{x}_d$. Then ϕ is continuous and $\pi_d = \phi \circ \pi_{d_1}$. Let $S : C_b(X_d, E) \to C_b(X_{d_1}, E)$ be the induced linear map. Then S is $\beta_p - \beta_p$ continuous. Let $v_d : C_b(X_d, E) \to \mathbb{K}, v_d(g) = \int g d\mu_d$ and $v_{d_1} : C_b(X_{d_1}, E) \to \mathbb{K}, v_{d_1}(g) = \int g d\mu_{d_1}$. Then $S^* v_{d_1} = v_d$. Since $< S^* v_{d_1}, h >=< v_{d_1}, Sh >=< v_{d_1}, h_1 >$, we get that $v_d(h) = v_{d_1}(h_1)$.
Let d_o be an ultrametric on E generating its topology. If $f \in C_b(X, E)$, then we have a continuous ultra-pseudometric on X defined by $d(x, y) = d_o(f(x), f(y))$ and f is d-continuous.

If $h : X_d \to E, h(\tilde{x}_d) = f(x)$, then h is well defined continuous and $T_d h = f$. Now we define u_m on $C_b(X, E)$ as follows: For $f \in C_b(X, E)$, choose a continuous ultra-pseudometric d on X such that $f = T_d g$ for some $g \in C_b(X_d, E)$. Define $u_m(f) = \int g d\mu_d$. As we have shown above, u_m is well defined and linear.
Claim II: u_m is $\beta_{e,p}$-continuous. In fact, let $W = \{f \in C_b(X, E) : |u_m(f)| \leq 1\}$ and let d be a continuous ultra-pseudometric on X. Then $V = \{g \in C_b(X_d, E) : |\int g d\mu_d| \leq 1\}$ is a β_p-neighborhood of zero in $C_b(X_d, E)$ and $T_d(V) \subset W$, which proves our claim.
Now there exists $m_1 \in \mathcal{M}_{\sigma,p}(X, E')$ such that $u_m(f) = \int f dm_1$ for each $f \in C_b(X, E)$. It is easy to see that $m_1(A)s = m(A)s$ for each clopen subset A of X and each $s \in E$ and so $m = m_1$. It follows that every $f \in C_b(X, E)$ is m-integrable and $u_m(f) = \int f dm$ and so u_m is $\beta_{e,p}$-continuous. This completes the proof.

From the preceding Theorem we get

Theorem 6.9 *If E is metrizable, then $\mathcal{M}_s(X, E')$ is algebraically isomorphic with the dual space of $(C_b(X, E), \beta'_e)$ via the isomorphism $m \mapsto u_m$.*

Theorem 6.10 *Assume that E is metrizable and let $m \in \mathcal{M}_{\sigma,p}(X, E')$. Then:*
(1) If $(A_i)_{i \in I}$ is a clopen partition of X, then, for each $\epsilon > 0$, the set $I_\epsilon = \{i \in I : m_p(A_i) \geq \epsilon\}$ is finite. Moreover, $m \in \mathcal{M}_{u,p}(X, E')$.
(2) $\mathcal{M}_{\sigma,p}(X, E') = \mathcal{M}_{u,p}(X, E')$.

Proof: (1) let $\lambda \in \mathbb{K}, |\lambda| > 1$. For each i, there exist a clopen subset B_i of A_i and $s_i \in E, p(s_i \leq 1$, such that $|m(B_i)s_i| \geq (2|\lambda|)^{-1} m_p(A_i)$. For $J \subset I$ finite, set $g_J = \sum_{i \in J} \chi_{B_i} s_i$. If $g = \sum_{i \in I} \chi_{B_i} s_i$, then $g_J \to g$ pointwise. The family $\{g_J : J \text{ finite}\}$ is τ_p-bounded and p-equicontinuous. In view of Theorem 4.6, we have that $\int g dm = \lim_J \int g_J dm = \sum_i m(B_i)s_i$. Given now $\epsilon > 0$, there exists a finite subset J of I such that $|m(B_i)s_i| < \frac{\epsilon}{2|\lambda|}$, for $i \notin J$, and hence $I_\epsilon \subset J$.
Let now B be a clopen subset of $\bigcup_{i \notin I_\epsilon} A_i$. For each $s \in E$, we have $m(B)s = \sum_{i \notin I_\epsilon} m(A_i \cap B)s$. Since, for $i \notin I_\epsilon$ and $p(s) \leq 1$, we have $|m(A_i \cap B)s| \leq \epsilon p(s)$, it follows that $m_p(D) \leq \epsilon$. By Theorem 5.5, $m \in \mathcal{M}_{u,p}(X, E')$.
(2) If $\mu \in \mathcal{M}_{u,p}(X, E')$, then the map u_μ is $\beta_{u,p}$-continuous and hence $\beta_{e,p}$-continuous, which implies that $\mu \in \mathcal{M}_{s,p}(X, E')$.

Corollary 6.11 *If E is metrizable, then $\mathcal{M}_s(X, E') = \mathcal{M}_u(X, E')$ is the common dual space of $C_b(X, E)$ under the topologies β'_e and β'_u.*

Theorem 6.12 *Let E be metrizable and let H be a subset of $\mathcal{M}_s(X, E')$. Then H is β'_u-equicontinuous iff it is β'_e-equicontinuous.*

Proof: Assume that H β'_e-equicontinuous. Then there exists $p \in cs(E)$ such that H is $\beta_{e,p}$-equicontinuous. Since $\beta_{e,p} \leq \tau_p$, it follows that $\sup_{m \in H} m_p(X) > \infty$. Let now $G \in \Omega_u$ and let $(A_i)_{i \in I}$ be a clopen partition of X such that every $\bar{A}_i^{\beta_o X}$ is disjoint from G. For each i, there exist a clopen subset B_i of A_i and $s_i \in E, p(s_i) \leq 1$, such that $|m(B_i)s_i| \geq (2|\lambda|)^{-1} \sup_{m \in H} m_p(A_i)$ (where $|\lambda| > 1$). For $J \subset I$ finite, set $h_J = \sum_{i \in J} \chi_{B_i} s_i$. The net $(h_J)_J$ is τ_p-bounded and p-equicontinuous. Moreover $h_J \to h = \sum_{i \in I} \chi_{B_i} s_i$ pointwise. Let μ be a non-zero element of $\mathbb{K}$. There exists a finite subset J_o of I such that $h - h_J \in \mu H^o$ if $J_o \subset J$. It follows that $\sup_{m \in H} m_p(A_i) \leq 2|\mu\lambda|$ if $i \notin J_o$. If $D = \bigcup_{i \notin J_o} A_i$, then we get that $m_p(D) \leq 2|\mu\lambda|$ for all $m \in H$. If now $r > 0$, then there exists $J \subset I$ finite such that $m_p(D) \leq 1/r$, for all $m \in H$, where $D = \bigcup_{i \notin J} A_i$. The set $A = \bigcup_{i \in J} A_i$ is clopen and its closure in $\beta_o X$ is disjoint from G. Moreover $\{f \in C_b(X,E) : \|f\|_p \leq r, \|f\|_{A,p} \leq 1/\alpha\} \subset H^o$, where $\alpha > \sup_{m \in H} m_p(X)$. This proves that H^o is a $\beta_{G,p}$-neighborhood of zero and so it is β'_u-equicontinuous. Since β'_u is coarser than β'_e, the result follows.

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